CHAPTER 1

MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.1 Matrices and Vectors

In many occasions, we can arrange a number of values of interest into an rectangular array. For example:

Example

	July	
Store	1	2
Newspaper	6	8
Magazines	15	20
Books	45	64

	August	
Store	1	2
Newspaper	7	9
Magazines	18	31
Books	52	68

We can represent the information on July sales more simply as

$$\begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix}.$$

$$\begin{bmatrix} 7 & 9 \\ 18 & 31 \\ 52 & 68 \end{bmatrix} \text{(Aug)}$$

elements of $\mathscr{R}$, the set of real numbers

Definitions

A **matrix** is a rectangular array of scalars.
If the matrix has m rows and n columns, we say that the **size** of the matrix is m **by** n, written $m \times n$.
The matrix is called **square** if $m = n$.
The scalar in the ith row and jth column is called the (i, j)**-entry** of the matrix.

Notation:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \in \mathcal{M}_{m\times n}$$

Example:

$$B = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix}$$

We use $\mathcal{M}_{m\times n}$ to denote the **set** that contains **all matrices whose sizes are** $m \times n$.

Equality of matrices

- equal: We say that two matrices A and B are **equal** if they have **the same size** and have **equal corresponding entries**.

Let $A, B \in \mathcal{M}_{m\times n}$.

Then $A = B \Leftrightarrow a_{ij} = b_{ij}, \forall i = 1, \ldots, m, j = 1, \ldots, n.$

Example

$A \neq B$ $\quad$ $A = C$ $\quad$ $A \neq D$

$$A = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 8 \\ 15 & 21 \\ 45 & 64 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \\ 17 & 38 \end{bmatrix}$$

Submatrices

- submatrix: A submatrix is obtained by deleting from a matrix entire rows and/or columns.
- For example,

$$E = \begin{bmatrix} 15 & 20 \\ 45 & 64 \end{bmatrix} \text{ is a submatrix of } B = \begin{bmatrix} \cancel{6} & \cancel{8} \\ 15 & 20 \\ 45 & 64 \end{bmatrix}.$$

Matrix addition

- Sum of matrices

Definition $A, B \in M_{m\times n}$

Let A and B be $m \times n$ matrices. We define the **sum** of A and B, denoted $A + B$, to be the $m \times n$ matrix obtained by adding the corresponding entries of A and B; that is, the $m \times n$ matrix whose (i, j)-entry is $a_{ij} + b_{ij}$.

Example

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} + \underset{B}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 8 \end{bmatrix}$$

Scalar multiplication

Definition

Let A be an $m \times n$ matrix and c be a scalar. The **scalar multiple** cA of the matrix A is defined to be the $m \times n$ matrix whose (i, j)-entry is ca_{ij}.

Example

$$\overset{c}{3} \cdot \overset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ -1 & 5 \end{bmatrix}} = \begin{bmatrix} 6 & 9 \\ 3 & 12 \\ -3 & 15 \end{bmatrix}$$

Zero matrices

- zero matrix: matrix with all zero entries, denoted by O (any size) or $O_{m\times n}$.

  For example, a 2-by-3 zero matrix can be denoted

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Property

$$A = O + A \text{ for all } A \in M_{m\times n}$$

Property

$$0 \cdot A = O \text{ for all } A$$

Question

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathcal{M}_{2\times 2}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in \mathcal{M}_{3\times 2}$.

Then both $0 \cdot A$ and $0 \cdot B$ can be denoted by $\underline{O}$, that is,

$$\underbrace{0 \cdot A = O}_{2\times 2}, \text{ and } \underbrace{0 \cdot B = O}_{3\times 2}.$$

Can we conclude that

$$\underline{0 \cdot A} \neq \underline{0 \cdot B}?$$

Matrix Subtraction

Definition

We define the matrix $-A$ to be $(-1)A$. The **matrix subtraction** of two matrices A and B is defined as

$$A - B = A + (-B).$$

Example

$$-\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ -1 & 5 \end{bmatrix} = (-1)\cdot\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -1 & -4 \\ 1 & -5 \end{bmatrix}$$

Question

For any $m \times n$ matrices A and B (i.e., $\forall A, B \in \mathcal{M}_{m\times n}$), will

$$A + B = B + A$$

always be true?

Question

For any $m \times n$ matrices A and B (i.e., $\forall A, B \in \mathcal{M}_{m\times n}$) and any real number s (i.e., $\forall s \in \mathcal{R}$), will

$$s(A + B) = sA + sB$$

always be true?

Question

For any $m \times n$ matrices A and B (i.e., $\forall A, B \in \mathcal{M}_{m\times n}$), will

$$A + B = B + A$$

always be true?

Answer: Yes! It is always true.

Proof:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & \cdot & a_{1n}+b_{1n} \\ & a_{ij}+b_{ij} & \\ a_{m1}+b_{m1} & \cdots & b_{mn}+b_{mn} \end{bmatrix} = [a_{ij}+b_{ij}] \qquad B + A = \begin{bmatrix} b_{11}+a_{11} & \cdots \\ & \\ & \end{bmatrix} = [b_{ij}+a_{ij}]$$

$$\forall i = 1, \cdots m, \forall j = 1, \cdots n, \quad a_{ij}+b_{ij} = b_{ij}+a_{ij}$$

Theorem 1.1 **(Properties of Matrix Addition and Scalar Multiplication)**

Let A, B, and C be $m \times n$ matrices, and let s and t be any scalars. Then

(a) $A + B = B + A$. (commutative law of matrix addition)

(b) $(A + B) + C = A + (B + C)$. (associative law of matrix addition)

(c) $A + O = A$.

(d) $A + (-A) = O$.

(e) $(st)A = s(tA)$.

(f) $s(A + B) = sA + sB$.

(g) $(s + t)A = sA + tA$.

Proof: All proofs can follow from basic arithmetic laws in $\mathcal{R}$ and previous definitions. Please do all of them yourself (homework).

By (b), sum of multiple matrices are written as $A + B + \cdots + M$

Transpose

Definition

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose (i, j)-entry is the (j, i)-entry of A.

Property

$$C \in \mathcal{M}_{m\times n} \Rightarrow C^T \in \mathcal{M}_{n\times m}$$

Example

$$C = \begin{bmatrix} 7 & 9 \\ 18 & 31 \\ 52 & 68 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 7 & 18 & 52 \\ 9 & 31 & 68 \end{bmatrix}$$

Question

Is $C = C^T$ always wrong?

Question

Is $\forall A, B \in \mathcal{M}_{m\times n}, (A+B)^T = A^T + B^T$ always true?

Theorem 1.2 (Properties of the Transpose)

Let A and B be $m \times n$ matrices, and let s be any scalar. Then

(a) $(A+B)^T = A^T + B^T$.

(b) $(sA)^T = sA^T$.

(c) $(A^T)^T = A$.

Proof: (b)

$$(sA)^T = \left(s \cdot \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\right)^T = \begin{bmatrix} sa_{11} & \cdots & sa_{1n} \\ \vdots & & \\ sa_{m1} & \cdots & sa_{mn} \end{bmatrix}^T$$

$$= \begin{bmatrix} sa_{11} & \cdots & sa_{m1} \\ \vdots & & \\ sa_{1n} & \cdots & sa_{mn} \end{bmatrix}$$

$$sA^T = s \cdot \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} sa_{11} & \cdots & sa_{m1} \\ \vdots & & \vdots \\ sa_{1n} & \cdots & sa_{mn} \end{bmatrix}$$

$\forall s \in \mathbb{R}$

$\forall A \in M_{m \times n}$

$$\Rightarrow (sA)^T = s \cdot A^T$$

Vectors

- A **row vector** is a matrix with one row.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

- A **column vector** is a matrix with one column.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$$

- The term **vector** can refer to either a **row vector** or a **column vector.**
- (Important) In this course, the term **vector** *always* refers to a **column vector** unless being explicitly mentioned otherwise.

Vectors

- $\mathcal{R}^n$:We denote the set of all **column vectors** with n entries by $\mathcal{R}^n$.
  - In other words,

$$\mathcal{R}^n = \mathcal{M}_{n\times 1}$$

- components: the entries of a vector.

Let $\mathbf{v} \in \mathcal{R}^n$ and assume

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \qquad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then **the ith component** of $\mathbf{v}$ refers to v_i.

Vector Addition and Scalar Multiplication

- Definitions of vector addition and scalar multiplication of vectors follow those for matrices.
- $\mathbf{0}$ is the zero vector (any size), and $\mathbf{u} + \mathbf{0} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in \mathcal{R}^n$.

A matrix is often regarded as a stack of **row vectors** or a cross list of **column vectors**. For any $C \in \mathcal{M}_{m\times n}$, we can write

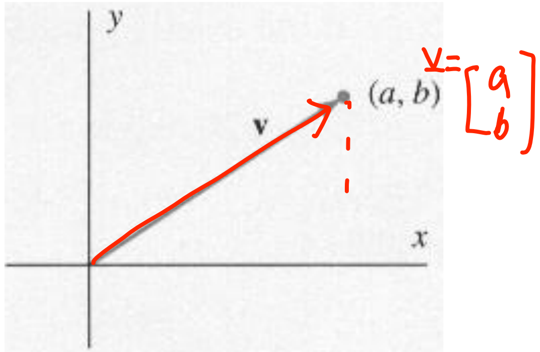
$$C = \underbrace{\begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_j & \cdots & \mathbf{c}_n \end{bmatrix}}_{m\times n}$$

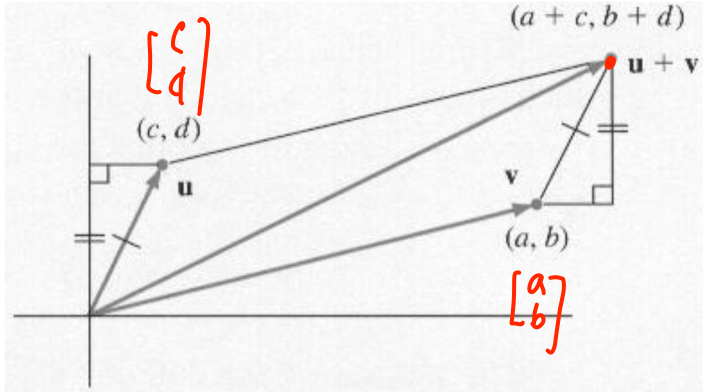
where $\mathbf{c}_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$

$$c_1 = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$$

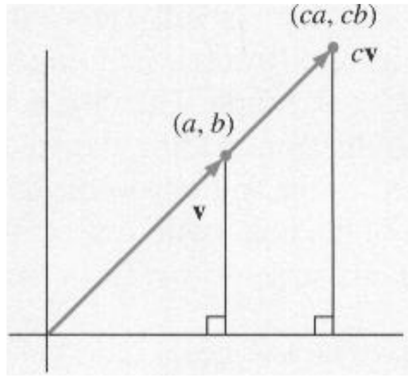
Geometrical Interpretations

Vectors for geometry in $\mathscr{R}^2$



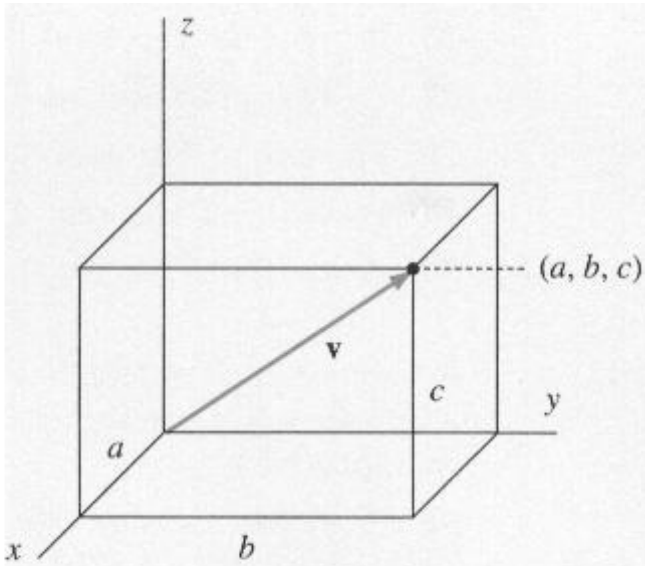


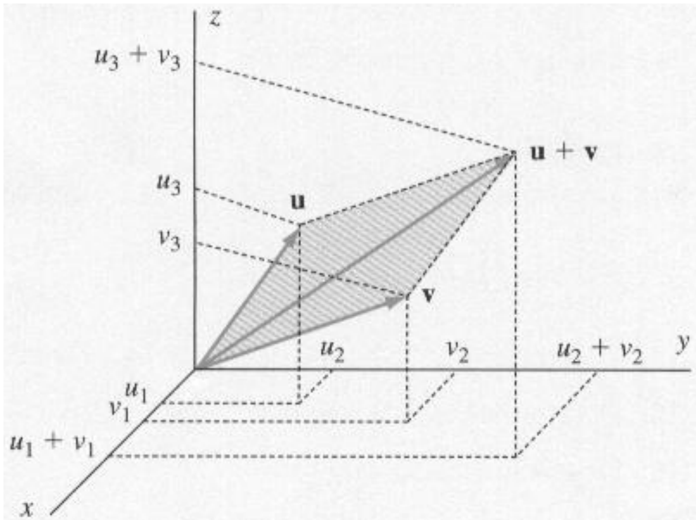
vector addition



scalar multiplication for a vector

in $\mathscr{R}^3$





Section 1.1 (Review)

- Matrix
  - Rows and columns
  - Size (m-by-n).
  - Square matrix.
  - (i,j)-entry
- Matrix
  - Equality,
  - Addition, Zero Matrix
  - Scalar multiplication, subtraction
- Vector
  - Row vectors, column vectors.
  - components

Definition

A **linear combination** of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is a vector of the form

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars. These scalars are called the **coefficients** of the linear combination.

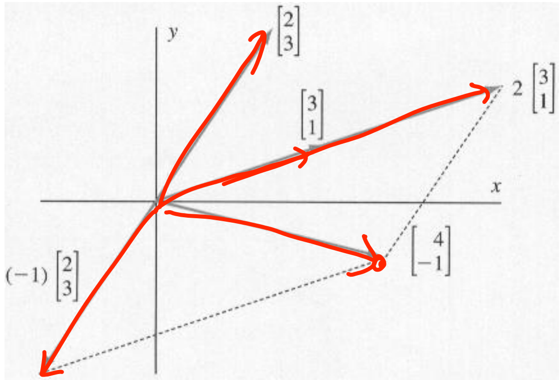
Example: $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Given the coefficients ({-3,4,1}), it is easy to compute the combination ($[2\ 8]^T$), but the inverse problem is harder.

Example: $\begin{bmatrix} 4 \\ -1 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 + x_2 \end{bmatrix}$

To determine x_1 and x_2, we must solve a system of linear equations, which has a unique solution $[x_1\ x_2]^T = [-1\ 2]^T$ in this case.

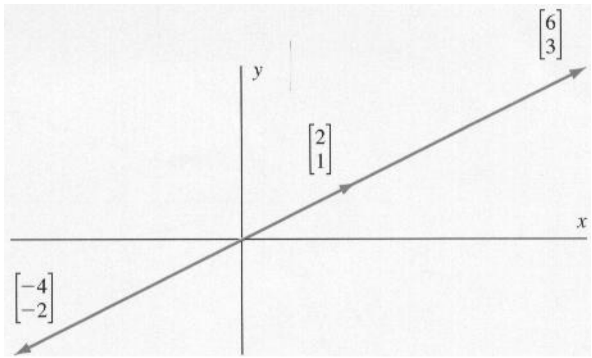
Geometrical view point:
manage to form a parallelogram



Example: to determine if [-4 -2]^T is a linear combination of [6 3]^T and [2 1]^T, we must solve

$$\begin{aligned} 6x_1 + 2x_2 &= -4 \\ 3x_1 + x_2 &= -2 \end{aligned}$$

which has infinitely many solutions, as the geometry suggests.



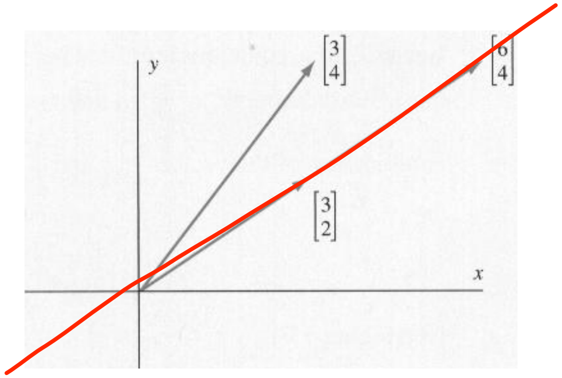
$$\begin{bmatrix}-4\\-2\end{bmatrix} = (-2)\begin{bmatrix}2\\1\end{bmatrix} + 0\cdot\begin{bmatrix}6\\3\end{bmatrix}$$

$$= 1\cdot\begin{bmatrix}2\\1\end{bmatrix} + (-1)\cdot\begin{bmatrix}6\\3\end{bmatrix}$$

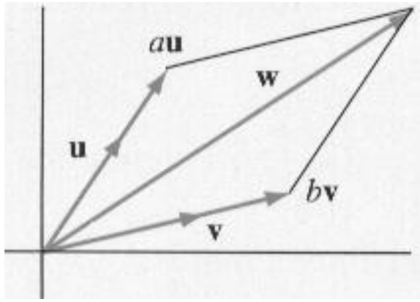
Example: to determine if $[3\ 4]^T$ is a linear combination of $[3\ 2]^T$ and $[6\ 4]^T$, must solve

$$\begin{aligned} 3x_1 + 6x_2 &= 3 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

which has no solutions, as the geometry suggests.



If **u** and **v** are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of **u** and **v** (unique linear combination).



algebraically, this means that **u** and **v** are nonzero vectors, and $\mathbf{u} \neq c\mathbf{v}$.

What is the condition in $\mathcal{R}^3$? in $\mathcal{R}^n$?

Standard vectors

The **standard vectors** of $\mathscr{R}^n$ are defined as

$$\underline{\mathbf{e}_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{\mathbf{e}_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \underline{\mathbf{e}_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Obviously, every vector in $\mathscr{R}^n$ may be uniquely linearly combined by these standard vectors.

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in R^n \qquad \underline{v} = v_1 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\underline{e_1}} + v_2 \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\underline{e_2}} + \cdots + v_n \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\underline{e_n}}$$

Matrix-Vector Product

Definition

Let A be an $m \times n$ matrix and $\mathbf{v}$ be an $n \times 1$ vector. We define the **matrix-vector product** of A and $\mathbf{v}$, denoted by $A\mathbf{v}$, to be the linear combination of the columns of A whose coefficients are the corresponding components of $\mathbf{v}$. That is,

$$A\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n.$$

Note that we can write: $A\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. Then $A\mathbf{v} = ?$

$$A\mathbf{v} = \underset{m \times n}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} 7 \\ 8 \end{bmatrix}} \underset{m \times 1}{=} 7\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 35 \end{bmatrix} + \begin{bmatrix} 16 \\ 32 \\ 48 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

Property: $A\mathbf{0} = \mathbf{0}$ and $O\mathbf{v} = \mathbf{0}$ for any A and $\mathbf{v}$.

Let $A \in \mathcal{M}_{2\times 3}$ and $\mathbf{v} \in \mathcal{R}^3$. Then

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + v_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \end{bmatrix}$$

More generally, when $A \in \mathcal{M}_{m\times n}$ and $\mathbf{v} \in \mathcal{R}^n$. Then

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1}\ \vdots & \vdots & & a_{in}\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \overset{a_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + v_2 \overset{a_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \cdots + v_3 \overset{a_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}$$

$$= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

The ith component of $A\mathbf{v}$ is $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Identity Matrix

Definition

For each positive integer n, the $n \times n$ **identity matrix** I_n is the $n \times n$ matrix whose respective columns are the standard vectors $\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_n$ in $\mathcal{R}^n$.

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sometimes I_n is simply written as I (any size).

Property: $I_n\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathcal{R}^n$

Stochastic Matrix

Definition

An $n \times n$ matrix $A \in \mathcal{M}_{n\times n}$ is called a **stochastic matrix** if all entries of A are nonnegative and the sum of all entries in each column is unity.

Example:

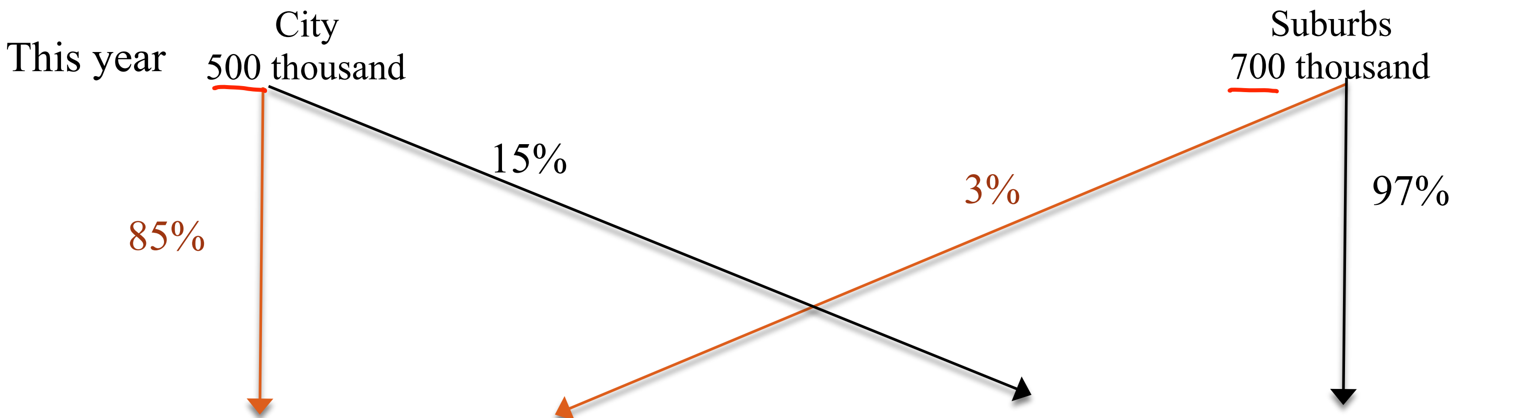
$$A = \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix} \text{ is a } 2 \times 2 \text{ stochastic matrix.}$$

Example: stochastic matrix

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc}\text{City} & \text{Suburbs}\end{array} \\ \text{To} & \begin{array}{r}\text{City}\\ \text{Suburbs}\end{array} & \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} = A \end{array}$$

probability matrix of a sample person's residence movement

$$\mathbf{p} = \begin{bmatrix} 500 \\ 700 \end{bmatrix}$$: current population of the city and suburbs



Next year $0.85 \times 500 + 0.03 * 700 = 446$ $0.15 \times 500 + 0.97 \times 700 = 754$

City Suburbs

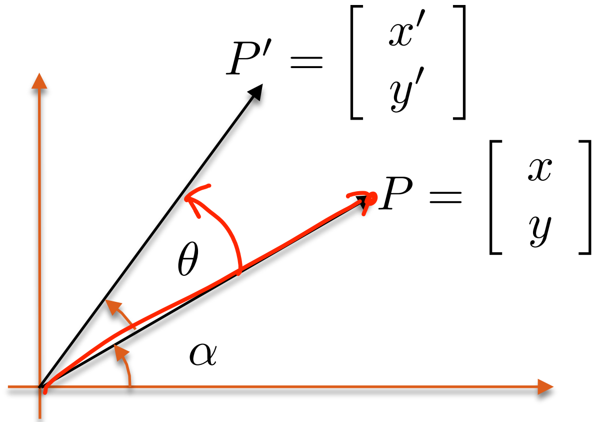
$$A\mathbf{p} = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \begin{bmatrix} 500 \\ 700 \end{bmatrix}$$: population distribution in the next year

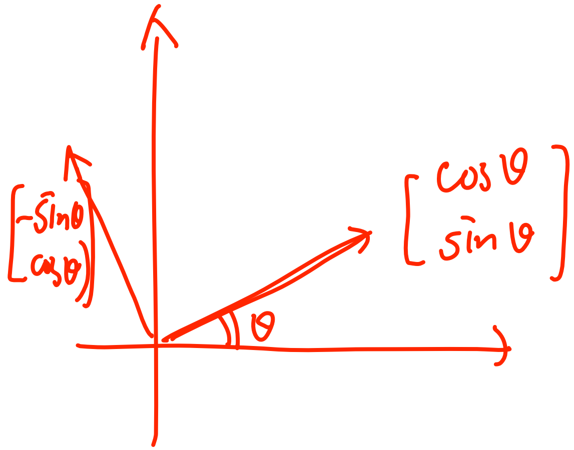
$A(A\mathbf{p})$: population distribution in the year following the next

Example: rotation matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} A_\theta \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \end{aligned}$$





Question

Is the statement

$$(A+B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}, \forall A, B \in \mathcal{M}_{m\times n}, \mathbf{u} \in \mathcal{R}^n$$

always true?

Question

Let $A \in \mathcal{M}_{m\times n}$ and $\mathbf{e}_j$ be **the jth standard vector in** $\mathcal{R}^n$. Then what is $A\mathbf{e}_j$?

$A = [\underline{a}_1 \ \ \underline{a}_2 \cdots \underline{a}_n]$

$A \cdot \underline{e}_j = [\underline{a}_1 \ \ \underline{a}_2 \cdots \underline{a}_n] \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th} = \underline{a}_j$

Theorem 1.3 (Properties of Matrix-Vector Products)

Let A and B be $m \times n$ matrices, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.

(b) $A(c\mathbf{u}) = c(A\mathbf{u}) = (cA)\mathbf{u}$ for every scalar c.

(c) $(A+B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$.

(d) $A\mathbf{e}_j = \mathbf{a}_j$ for $j = 1, 2, \ldots, n$, where $\mathbf{e}_j$ is the jth standard vector in $\mathcal{R}^n$.

(e) If B is an $m \times n$ matrix such that $B\mathbf{w} = A\mathbf{w}$ for all $\mathbf{w}$ in $\mathcal{R}^n$, then $B = A$.

(f) $A\mathbf{0}$ is the $m \times 1$ zero vector.

(g) If O is the $m \times n$ zero matrix, then $O\mathbf{v}$ is the $m \times 1$ zero vector.

(h) $I_n\mathbf{v} = \mathbf{v}$.

Proof for (e): If $B \neq A$, then $(B - A)\mathbf{e}_j \neq \mathbf{0}$, i.e., $B\mathbf{e}_j \neq A\mathbf{e}_j$, for some j.

$$\overset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problems for practice (1.1~1.2)

Section 1.1: Problems 1, 5, 7, 11, 13, 19, 25, 27, 29, 31, 37, 39, 41, 43, 45, 51, 53, 55.

Section 1.2: Problems 3, 5, 8, 9, 15, 34, 42, 44, 83-87